MECHANICS BY QUATERNIONS.

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(21). *Center of Gravity.*—WE have seen (Eq. 21) that the vector of the center of mass of a system of particles whose weights are w_1, w_2, w_3, etc., is given by the equation

$$,\rho_0 = \frac{\Sigma(w,\rho)}{\Sigma(w)}$$

To apply this formula to finding the center of gravity of any line, surface, or volume we have only to substitute for w the element of the line, surface, or volume multiplied by a coefficient of heaviness, aud then to substitute integration for summation. Let h be the weight of a unit of volume of the substance. If it is the same at all points of the body it may be taken outside of the sign of integration, and will then cancel out; if it is *not* constant it must be expressed as a function of $,\rho$ before the integration can be performed.

(22). *Center of Gravity of a Line.*—By this is meant the c. g. of such a body as a fine wire bent into some curve, the diameter being so small compared with the length that it may be regarded for practical purposes as a true mathematical line, i. e., as if the wire, still retaining its weight, should have its diameter reduced to nothing. Let the equation of the curve be

$$,\rho = ,\varphi(t). \quad \therefore d,\rho = ,\varphi'(t)dt,$$

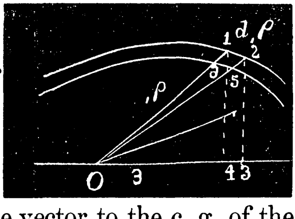
and $ds = Td,\rho = T,\varphi'(t).dt =$ element of curve. Therefore, for this case,

$$,\rho_0 = \frac{\int,\rho hTd,\rho}{\int hTd,\rho} = \frac{\int h,\varphi(t)T,\varphi'(t).dt}{\int hT,\varphi'(t).dt} \qquad (56)$$

(23) *Center of Gravity of a Plane Area.*—Let the bounding curve be

$$,\rho = ,\varphi(t),$$

and let ε be some unit vector in the plane of the curve. We may take either 1 2 3 4 as the element of area or 1 2 5 6. In the first case we have area 1 2 3 4 $= TV\varepsilon,\rho S\varepsilon^{-1}d,\rho$ (at the limit); so that the area between a pair of perpendiculars to ε will be



$$\int TV\varepsilon,\rho S\varepsilon^{-1}d,\rho = \int TV\varepsilon,\varphi S\varepsilon^{-1},\varphi'.dt,$$

where for brevity $,\varphi$ is written for $,\varphi(t)$. Now the vector to the c. g. of the element is $,\rho + \frac{1}{2}\varepsilon V\varepsilon,\rho = ,\varphi + \frac{1}{2}\varepsilon V\varepsilon,\varphi$; hence for the center of gravity of the area between ε and the curve

$$,\rho_0 = \int h(,\varphi+\tfrac{1}{2}\varepsilon V\varepsilon,\varphi)TV\varepsilon,\varphi S\varepsilon^{-1},\varphi'.dt \div \int hTV\varepsilon,\varphi S\varepsilon^{-1},\varphi'.dt. \qquad (57)$$

In order to get the element 1 2 5 6 we must evidently change the tensor of ${}_{\prime}\rho$ independently of the versor. Therefore write the equation ${}_{\prime}\rho = u{}_{\prime}\varphi(t)$. By giving u a succession of values we shall obtain a series of curves such as 5 6. If we differentiate with respect to u only $T{}_{\prime}\rho$ varies, so that the end of ${}_{\prime}\rho$ moves say from 1 to 6, while if we differentiate with respect to t the end of ${}_{\prime}\rho$ moves along such a curve as 1 2 or 6 5. Let

$$\frac{d{}_{\prime}\rho}{dt} = D_t \text{ and } \frac{d{}_{\prime}\rho}{du} = D_u,$$

then the area of element 1 2 5 6 will be $TVD_uD_t.du\,dt$. But $D_u = {}_{\prime}\varphi$, and $D_t = u{}_{\prime}\varphi'$, by the equation assumed above. Therefore

$${}_{\prime}\rho_0 = \int\int h{}_{\prime}\varphi TV{}_{\prime}\varphi{}_{\prime}\varphi'.u^2dt\,du \div \int\int hTV{}_{\prime}\varphi{}_{\prime}\varphi'.u\,dt\,du; \qquad (58)$$

or integrating for u from 0 to 1, to cover the space from the origin to the bounding curve,

$${}_{\prime}\rho_0 = \tfrac{1}{3}\int h{}_{\prime}\varphi TV{}_{\prime}\varphi{}_{\prime}\varphi'.dt \div \tfrac{1}{2}\int hTV{}_{\prime}\varphi{}_{\prime}\varphi'.dt. \qquad (59)$$

This integration however can only be performed when h is *not a function* of u.

Equations (58) and (59) hold equally well when ${}_{\prime}\rho = {}_{\prime}\varphi(t)$ does not represent a plane curve. They give in any case the c. g. of the surface swept over by the radius vector.

(24). *Any Surface.*—Let the the equation of the surface be ${}_{\prime}\rho = {}_{\prime}\varphi(x,y)$. Then $D_x = \dfrac{d{}_{\prime}\rho}{dx}$, and $D_y = \dfrac{d{}_{\prime}\rho}{dy}$ are vectors || to tangents to the surface at the end of ${}_{\prime}\rho$, for by supposing x to be constant we have one curve on the surface, and by supposing y to be constant we have another. By giving successive constant values to x and y the surface will be divided up into four-sided figures which when small enough may be regarded as parallelograms. The area of such an elem. parallelogram will be $TVD_xD_y.dx\,dy$. ∴

$${}_{\prime}\rho_0 = \int\int h{}_{\prime}\rho TVD_xD_y.dx\,dy \div \int\int hTVD_xD_y.dx\,dy. \qquad (60)$$

(25) *Any Solid.*—If the equation of the bounding surface is ${}_{\prime}\rho = {}_{\prime}\varphi(x,y)$, then to obtain a parallelopipedical element we must vary the tensor of ${}_{\prime}\rho$. Therefore write, as in Art. 23, ${}_{\prime}\rho = u{}_{\prime}\varphi(x, y)$. By giving u successive values differing by du, and extending from 0 to 1 we divide up the solid into a series of shells. The solid element will then be $SD_uD_xD_y.du\,dx\,dy$. But

$$D_u = {}_{\prime}\varphi,\ D_x = u\frac{d{}_{\prime}\varphi}{dx},\ D_y = u\frac{d{}_{\prime}\varphi}{dy};\ \therefore$$

$${}_{\prime}\rho_0 = \int\int\int hu^3{}_{\prime}\varphi S{}_{\prime}\varphi\frac{d{}_{\prime}\varphi}{dx}\frac{d{}_{\prime}\psi}{dy}.dx\,dy\,du \div \int\int\int hu^2S{}_{\prime}\varphi\frac{d{}_{\prime}\varphi}{dx}\frac{d{}_{\prime}\varphi}{dy}.dxdydu. (61)$$

If h is not a function of u we can integrate for u between 0 and 1, thus obtaining

$$,\psi_0 = \tfrac{1}{4}\int\int h,\varphi S,\varphi \frac{d,\varphi}{dx}\frac{d,\varphi}{dy}.dx\,dy \div \tfrac{1}{3}\int\int hS,\varphi\frac{d,\varphi}{dx}\frac{d,\varphi}{dy}.dx\,dy. \qquad (62)$$

(26). *Surface of Revolution.*—We will now apply eq. (60) to the general equation of a surface of revolution. Let $,\rho = ,\varphi(t)$ be the equation of any curve, plane or tortuous. If this curve be revolved about an axis along which ε is a unit vector, and the origin be taken at a point of this axis, a surface of revolution will be generated whose equation will be

$$,\rho = \varepsilon^{\frac{\theta}{\pi}}\,,\varphi(t)\,\varepsilon^{-\frac{\theta}{\pi}}\,* \qquad (63)$$

This will appear from the fact that if θ be constant in eq. (63), the eq'n represents the generating curve turned through the angle θ about ε, while if t be constant the equation is that of a circle generated by the extremity of $,\varphi(t)$ revolving about ε.

To apply (60) we have to evaluate TVD_xD_y.

$$D_x = D\theta = \varepsilon^{\frac{2\theta}{\pi}}\,V\varepsilon,\varphi \text{ and } D_y = D_t = \varepsilon^{\frac{\theta}{\pi}}\,,\varphi'\varepsilon^{-\frac{\theta}{\pi}},$$

in which the t is omitted for brevity.

$$\begin{aligned}\therefore\ TVD_xD_y &= TVD\theta\,Dt = TV.\varepsilon^{2\theta\div\pi}\,V\varepsilon,\varphi.\varepsilon^{\theta\div\pi},\varphi'\varepsilon^{-\theta\div\pi}\\ &= TV[\varepsilon\,V\varepsilon,\varphi S\varepsilon,\varphi'\cos\theta - V\varepsilon,\varphi S\varepsilon,\varphi'\sin\theta + \varepsilon S.\varepsilon,\varphi'\,V\varepsilon,\varphi]\\ &= \sqrt{(S^2.\varepsilon,\varphi'\,V\varepsilon,\varphi - V^2\varepsilon,\varphi S^2\varepsilon,\varphi')} = \sqrt{(,\varphi'^2\,V^2\varepsilon,\varphi - S^2\varepsilon,\varphi,\varphi')}\\ &= TV.,\varphi'\,V\varepsilon,\varphi. \text{ Therefore (60) becomes}\end{aligned}$$

$$,\rho_0 = \int\int h\,\varepsilon^{\theta\div\pi},\varphi\,\varepsilon^{-\theta\div\pi}\,TV.,\varphi'\,V\varepsilon,\varphi\,dt\,d\theta \div \int\int hTV.,\varphi'\,V\varepsilon,\varphi\,dt\,d\theta. (64)$$

If h be not a function of θ we may integrate for θ from 0 to θ_1, obtaining

$$,\rho_0 = \int h[\theta_1\varepsilon^{-1}S\varepsilon,\varphi - \left(\varepsilon^{2\theta_1\div\pi} - 1\right)V\varepsilon,\varphi]TV.,\varphi'\,V\varepsilon,\varphi dt \div \theta_1\int hTV.,\varphi'\,V\varepsilon,\varphi dt \qquad (65)$$

If the integration be for a complete revolution so that $\theta_1 = 2\pi$, then the equation reduces to

$$,\rho_0 = \int h\varepsilon^{-1}S\varepsilon,\varphi TV.,\varphi'\,V\varepsilon,\varphi\,dt \div \int hTV.,\varphi'\,V\varepsilon,\varphi\,dt. \qquad (66)$$

If the generating curve lie in a plane passing through the axis we shall have $S\varepsilon,\varphi,\varphi' = 0$, so that in this case $TV.,\varphi'\,V\varepsilon,\varphi = T,\varphi'\,V\varepsilon,\varphi$.

(27). *Solid of Revolution.*—For the solid generated by the radius vector, when its extremity generates the surface of eq. (63), we have as in Art. 25,

$$,\rho = u\varepsilon^{\theta\div\pi},\varphi(t)\varepsilon^{-\theta\div\pi}.$$

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*See Tait's Quaternions, Art. 351

We need these to evaluate

$$SD_uD\theta D_t = SD_u VD\theta D_t = u^2 S\varepsilon^{\frac{\theta}{\pi}}{}_{,}\varphi\varepsilon^{-\frac{\theta}{\pi}} V.\varepsilon^{\frac{2\theta}{\pi}} V\varepsilon_{,}\varphi'.\varepsilon^{\frac{\theta}{\pi}}{}_{,}\varphi\varepsilon^{-\frac{\theta}{\pi}},$$

or by the value of $VD\theta D_t$ of the last Art.

$$SD_uD\theta D_t = u^2 S(\varepsilon^{-1}S\varepsilon_{,}\varphi + \varepsilon^{-1}V\varepsilon_{,}\varphi\cos\theta + V\varepsilon_{,}\varphi\sin\theta)$$
$$\times(\varepsilon V\varepsilon_{,}\varphi S\varepsilon_{,}\varphi'\cos\theta - V\varepsilon_{,}\varphi S\varepsilon_{,}\varphi'\sin\theta + \varepsilon S.\varepsilon_{,}\varphi' V\varepsilon_{,}\varphi)$$
$$= u^2[S.\varepsilon_{,}\varphi' V\varepsilon_{,}\varphi S\varepsilon_{,}\varphi - V^2\varepsilon_{,}\varphi S\varepsilon_{,}\varphi'] = -S._{,}\varphi_{,}\varphi' V\varepsilon_{,}\varphi.u^2.$$

Hence substituting this value and that of $_{,}\rho$ above in eq. (61) we have

$$_{,}\rho_0 = \iiint hu^3\varepsilon^{\frac{th}{\pi}}{}_{,}\phi\varepsilon^{-\frac{th}{\pi}} S._{,}\phi_{,}\phi' V\varepsilon_{,}\phi.dt\,d\theta\,du \div \iiint hu^2 S._{,}\phi_{,}\phi' V\varepsilon_{,}\phi.dt\,d\theta\,du. \quad (67)$$

[Here, and through the remainder of this paper, for want of sorts, ϕ is substituted for φ and for subscripts and indices th is written for θ.]

If h be not a function of θ we can integrate as in last article from 0 to θ_1, obtaining

$$_{,}\rho_0 = \iint hu^3\left[\theta_1\varepsilon^{-1}S\varepsilon_{,}\phi - \left(\varepsilon^{\frac{2th}{\pi}} - 1\right)V\varepsilon_{,}\phi\right]S._{,}\phi_{,}\varphi' V\varepsilon_{,}\phi.dt\,du$$
$$\div\theta_1\iint hu^2 S._{,}\phi_{,}\phi' V\varepsilon_{,}\phi.dt\,du. \quad (68)$$

If $\theta_1 = 2\pi$, for a complete revolution,

$$_{,}\rho_0 = \varepsilon^{-1}\iint hu^3 S\varepsilon_{,}\phi S._{,}\varphi_{,}\phi' V\varepsilon_{,}\phi.dt\,du \div \iint hu^2 S._{,}\phi_{,}\phi' V\varepsilon_{,}\phi.dt\,du. \quad (69)$$

If the generating curve be in a *plane through the axis*, so that $V.V_{,}\phi_{,}\phi'$-$V\varepsilon_{,}\phi = 0$, we may replace the quantity $S._{,}\phi_{,}\phi' V\varepsilon_{,}\phi$ in eqs. (67), (68) and (69) by the equivalent expression $TV_{,}\phi_{,}\phi' V\varepsilon_{,}\phi$. For with the above condition we have

$$V^2{}_{,}\phi_{,}\phi' V^2\varepsilon_{,}\phi = T^2 V_{,}\phi_{,}\phi' V\varepsilon_{,}\phi = S^2._{,}\phi_{,}\phi' V\varepsilon_{,}\phi;\ \therefore\ TV_{,}\phi_{,}\phi' V\varepsilon_{,}\phi$$
$$= \pm S._{,}\phi_{,}\phi' V\varepsilon_{,}\phi.$$

(28). *Guldinus' Properties.*—We have by Art. 26, the area of a surface of revolution formed by revolving a plane curve about an axis in its plane through an angle $2\pi\div n$,

$$\mathrm{S} = \frac{2\pi}{n}\int T_{,}\phi' V\varepsilon_{,}\phi.dt:$$

also by eq. (56) omitting the h, i. e., supposing the density uniform,

$$_{,}\rho_0 = \int {}_{,}\phi T_{,}\phi'.dt \div \int T_{,}\phi'.dt.$$

Multiply both sides of this eqn. by $(2\pi\div n)\int T_{,}\phi'.dt$ and operate by $TV\varepsilon$;

$$\therefore\ \frac{2\pi}{n}TV\varepsilon_{,}\rho_0.\int T_{,}\phi' dt = \frac{2\pi}{n}\int T_{,}\phi' V\varepsilon_{,}\phi.dt = \mathrm{S}. \quad (70)$$

The factor outside the sign of integration in the first member is the nth part of the circumf of the circle in which the c. g. of the curve $_{,}\rho = {}_{,}\phi(t)$

moves, while the integral in that member is the length of the same curve; hence the area of the surface generated by a plane curve revolving about an axis in its own plane is equal to the length of the curve multiplied by the length of the path of the center of gravity of the curve.

Again, the volume generated by revolving the area bounded by a plane curve about an axis in its own plane through an angle $2\pi \div n$ is, by Art. 27,

$$V = \frac{2\pi}{n}\int\int TV{}_{,}\phi{}_{,}\phi' \, V\varepsilon{}_{,}\phi . u^2 du\, dt,$$

and by eq. (58), omitting h as before, the c. g. of a plane area is given by

$$_{,}\rho_0 = \frac{2\pi}{n}\int\int {}_{,}\phi \, TV{}_{,}\phi{}_{,}\phi' . u^2 du\, dt \div \frac{2\pi}{n}\int\int TV{}_{,}\phi{}_{,}\phi' . u\, du\, dt.$$

or clearing and operating by $TV.\varepsilon$

$$\frac{2\pi}{n} TV\varepsilon{}_{,}\rho_0 . \int\int TV{}_{,}\phi{}_{,}\phi' . u\, du\, dt = \frac{2\pi}{n}\int\int TV{}_{,}\phi{}_{,}\phi' \, V\varepsilon{}_{,}\phi . u^2 du\, dt \quad (71)$$
$$= V.$$

Hence the volume of the solid formed by revolving a plane area about an axis in its own plane is equal to the plane area multiplied by the length of the path of the center of gravity.

(29) These properties may be extended to the cases of a plane curve or a plane area moving so that the center of gravity of the curve or area generates some curve $_{,}\rho = {}_{,}\phi(t)$ to which the plane of the curve or area is always normal. Let l be the length of the moving curve and A its area, S and V respectively the surface and volume generated and $_{,}\rho = {}_{,}\phi(t)$ the path of the center of gravity. Then for the length of an elementary arc $d_{,}\rho$ the curve will coincide with the arc of its osculating circle for that point, and therefore Guldinus' properties will hold; hence $dS = lTd_{,}\rho = lT{}_{,}\phi' . dt$, and $dV = AT{}_{,}\phi' . dt$; and by integration

$$S = l\int T{}_{,}\phi' . dt; \qquad V = A\int T{}_{,}\phi' . dt. \quad (72)$$

(30). A few examples to illustrate the application of these formulæ will now be given

Center of Gravity of the arc of a Cycloid.—If θ be the angle turned thro' by the rolling circle, and a the radius of this circle; then the equation of the cycloid with the origin at a cusp may be written

$$_{,}\rho = {}_{,}\phi(\theta) = ai(\theta - \sin\theta) + aj(1 - \cos\theta),$$

in which i is a unit vector along the base, and j a unit vector along the tangent at the origin. Therefore $d_{,}\rho \div d\theta = {}_{,}\phi'(\theta) = ai(1 - \cos\theta) + aj\sin\theta$, and $T{}_{,}\phi'(\theta) = a\sqrt{[2(1 - \cos\theta)]} = 2a\sin\frac{1}{2}\theta$. Therefore, by eq. (56), putting θ for t, and supposing h constant,

$$_{,}\rho_0 = a\int [i(\theta - \sin\theta) + j(1 - \cos\theta)]\sin\tfrac{1}{2}\theta . d\theta \div \int \sin\tfrac{1}{2}\theta . d\theta.$$

By integrating this equation from 0 to π, i. e. from the cusp to the vertex of the curve, we have ${}_{\prime}\rho_0 = \frac{4}{3}a(i+j)$. If we go from the origin to the next cusp, the limits will be 0 and 2π, and ${}_{\prime}\rho_0 = a(\pi i+\frac{4}{3}j)$.

As another example take the tortuous curve whose equation is

$$ {}_{\prime}\rho = a(it+\tfrac{1}{2}\sqrt{2}.j\,t^2+\tfrac{1}{3}kt^3), $$

and which is projected on the three reference planes into a common, cubic, and semi-cubic parabola respectively. $d_{\prime}\rho\div dt = {}_{\prime}\phi'(t) = a(i+jt\sqrt{2}+kt^2)$;

$$ \therefore\ T_{\prime}\phi'(t) = a(1+t^2). $$

$$ \therefore\ {}_{\prime}\rho_0 = a\int(it+\tfrac{1}{2}jt^2\sqrt{2}+\tfrac{1}{3}kt^3)(1+t^2)dt\div\int(1+t^2)dt. $$

If the integration be from 0 to 1 we shall have

$$ {}_{\prime}\rho_0 = \tfrac{9}{16}i+\tfrac{1}{5}\sqrt{2}.j+\tfrac{5}{48}k. $$

For an example of the application of eq. (57) take the parabola whose equation is ${}_{\prime}\rho = {}_{\prime}at+\varepsilon\frac{1}{2}t^2$, in which ε is a unit vector along the axis, and ${}_{\prime}a$ is a vector along the tangent at the vertex, so that $S\varepsilon_{\prime}a = 0$.

$\therefore\ {}_{\prime}\phi'(t) = {}_{\prime}a + \varepsilon t,\ V\varepsilon_{\prime}\phi = tV\varepsilon_{\prime}a = t\varepsilon_{\prime}a,\ \varepsilon V\varepsilon_{\prime}\phi = -t_{\prime}a,\ TV\varepsilon_{\prime}\phi = at$ and $S\varepsilon^{-1}{}_{\prime}\phi' = t$. Suppose h to be constant, then

$$ {}_{\prime}\rho_0 = \int_0^t \tfrac{1}{2}({}_{\prime}at+\varepsilon t^2)at^2.dt\div\int_0^t at^2dt = \frac{\frac{1}{4}t^4{}_{\prime}a+\frac{1}{5}\varepsilon t^5}{\frac{2}{3}t^3} $$

$$ = \tfrac{3}{8}t_{\prime}a+\tfrac{3}{10}t^2\varepsilon = \tfrac{3}{8}{}_{\prime}at+\tfrac{3}{5}\varepsilon\tfrac{1}{2}t^2. $$

Let us apply eq. (59) to the equation of the parabola just used, now however regarding ${}_{\prime}a$ as a vector along *any* tangent and ε as a unit vector along the diameter through the point of contact of the same. Then

$$ TV_{\prime}\phi_{\prime}\phi' = tTV({}_{\prime}a + \tfrac{1}{2}\varepsilon t)({}_{\prime}a + \varepsilon t) = t^2TV({}_{\prime}a\varepsilon - \tfrac{1}{2}{}_{\prime}a\varepsilon) = \tfrac{1}{2}t^2TV_{\prime}a\varepsilon. $$

$$ \therefore\ {}_{\prime}\rho_0 = \tfrac{2}{3}\int_9^t({}_{\prime}a+\tfrac{1}{2}\varepsilon t)\,t^3dt\div\int_0^t t^2dt = \tfrac{1}{2}{}_{\prime}at+\tfrac{4}{5}\varepsilon\tfrac{1}{2}t^2. $$

This gives the c. g. of any segment of a parabola.

The equation of a circle may be written ${}_{\prime}\rho = {}_{\prime}\phi(\theta) = a(i\cos\theta+j\sin\theta)$; $\therefore\ {}_{\prime}\phi' = a(-i\sin\theta+j\cos\theta)$, and $TV_{\prime}\phi_{\prime}\phi' = a^2$. Therefore, by eq. (58), if $h =$ constant, and $\theta = t$,

$$ {}_{\prime}\rho_0 = a\int_{-th}^{+th}\int_u^1(i\cos\theta+j\sin\theta)\,u^2d\theta\,du\div\int_{-th}^{+th}\int_u^1 ud\theta du $$

$$ = \tfrac{1}{3}a(1-u^3)\big[i\sin\theta-j\cos\theta\big]_{-th}^{+th}\div\tfrac{1}{2}(1-u^2)\big[\theta\big]_{-th}^{+th} $$

$$ = \frac{2}{3}\cdot\frac{1-u^3}{1-u^2}\cdot\frac{i\sin\theta}{\theta}.a. $$

This gives the c. g. of a portion of a ring contained bet. two concentric cir.

$$ \frac{1-u^3}{1-u^2} = \frac{(1-u)(1+u+n^2)}{(1-u)(1+u)} = \frac{1+u+u^2}{1+u};\ \therefore\ {}_{\prime}\rho_0 = \frac{2}{3}\cdot\frac{1+u+u^2}{1+u}\ \frac{\sin\theta}{\theta}.ia. $$

If $u = 1$, we have for an arc of a circle, ${}_{\prime}\rho_0 = (\sin\theta\div\theta).ia$.

Ellipsoid.—The equation of this surface may be written

$$_{\prime}\rho = {}_{\prime}a\cos x + {}_{\prime}b\sin x\cos y + c\sin x\sin y,$$

in which a, b, c are the semi-axes. The c. g. of the solid ellipsoidal shell will be found first by eq. (61), and then that of the surface, from this by making the thickness of the shell infinitesimal, for the reason that the scalar in eq, (61) is much simpler than the tensor in eq. (60) in the case of the ellipsoid. We have then

$$\frac{d_{\prime}\phi}{dx} = -{}_{\prime}a\sin x + {}_{\prime}b\cos x\cos y + {}_{\prime}c\cos x\sin y,$$

$$\frac{d_{\prime}\phi}{dy} = -{}_{\prime}b\sin x\sin y + {}_{\prime}c\sin x\cos y. \quad \text{Whence}$$

$$S_{\prime}\phi\frac{d_{\prime}\phi}{dx}\frac{d_{\prime}\phi}{dy} = S_{\prime}a_{\prime}b_{\prime}c\sin x = -abc\sin x.$$

Thus eq. (61) becomes, if h is constant as usual,

$$_{\prime}\rho_0 = \int_0^x\int_0^y\int_u^1({}_{\prime}a\cos x + {}_{\prime}b\sin x\cos y + {}_{\prime}c\sin x\sin y)u^3\sin x.dx\,dy\,du$$
$$\div\int_0^x\int_0^y\int_u^1 u^2\sin x.dx\,dy\,du,$$

in which equation the limits are so taken as to include a shell bounded by two similar ellipsoids, the plane $_{\prime}a_{\prime}b$, a plane through $_{\prime}a$ inclined to the plane of $_{\prime}a$ and $_{\prime}b$ at an angle $\tan^{-1}[(c\div b)\tan y]$, and a cone with its vertex at the origin and a section of the ellipsoid by a plane || to the plane of $_{\prime}b$ and $_{\prime}c$ for a directrix, the distance of this last plane from the origin being $_{\prime}a\cos x$. By integration we find

$$_{\prime}\rho_0 = \frac{3}{4}\cdot\frac{1-u^4}{1-u^3}\cdot\frac{{}_{\prime}a\,y\sin^2x + {}_{\prime}b\sin y\,(x-\sin x\cos x) + {}_{\prime}c(1-\cos y)\,(x-\sin x\cos x)}{2y(1-\cos x)}$$

For one-eighth part of the ellipsoidal shell $x = y = \frac{1}{2}\pi$, and

$$_{\prime}\rho_0 = \frac{3}{8}\left(\frac{1-u^4}{1-u^2}\right)({}_{\prime}a + {}_{\prime}b + {}_{\prime}v).$$

For one-fourth $x = \pi$, $y = \frac{1}{2}\pi$,

$$\therefore {}_{\prime}\rho_0 = \frac{3}{8}\left(\frac{1-u^4}{1-u^3}\right)({}_{\prime}b + {}_{\prime}c).$$

If we had integrated from $-y$ to $+y$ instead of from 0 to y the result above would have been unchanged except that the term containing c would not appear; if then x be taken from 0 to π we shall have a lune lying between two planes through $_{\prime}a$ making equal angles with the plane of $_{\prime}a$ and $_{\prime}b$, of which the c. g. will be given by

$$_{\prime}\rho_0 = \frac{3\pi_{\prime}b}{16}\cdot\frac{\sin y}{y}\cdot\frac{1-u^4}{1-u^3}.$$

To obtain the solids reaching to the center we have only to make $u = 0$ in each of the above equations. For the surface of the ellipsoid we have

$$\frac{1-u^4}{1-u^3} = \frac{1+u+u^2+u^3}{1+u+u^2} = \frac{4}{3},$$

where $u = 1$; and this value substituted in each of the above expressions will give the desired results. Of course we have only to make $a = b = c$ to get the corresponding results for the sphere.

As an example of the use of (64) let us apply it to the torus generated by the revolution of the circle $_{\prime}\rho = bj + a(i \cos t + j \sin t)$ about the axis of i.

We shall have then $\varepsilon = i$ and for the equation of the torus

$$_{\prime}\rho = i^{th\div\pi}[ai \cos t + (b + a \sin t)j]i^{-th\div\pi}.$$

$V\varepsilon_{\prime}\phi = Vi[ai \cos t + (b + a \sin t)j] = (b+a \sin t)k$, $_{\prime}\phi'(t) = a(-i \sin t + j \cos t)$, $T_{\prime}\phi' V\varepsilon_{\prime}\phi = a(b+a \sin t)$, $S\varepsilon_{\prime}\varphi = Si_{\prime}\phi = -a \cos t$. Therefore

$$_{\prime}\rho_0 = \int_{+t}^{\pi-t}\left[ia\theta \cos t - i^{\frac{2th}{\pi}} k(b+a \sin t)\right]_{-th}^{+th}(b+a \sin t)\, dt$$
$$\div 2\theta\int_{+t}^{\pi-t}(b+a \sin t)\, dt,$$

The limits are taken from $-\theta$ to $+\theta$ and from t to $\pi - t$ so as to get a portion of the surface symmetrical about j. From this we find on integr'n,

$$_{\prime\prime}\rho_0 = \frac{[b^2(\pi-2t)+4ab \cos t+\frac{1}{2}a^2(\pi-2t-2 \sin t \cos t)]j \sin\theta}{\theta(b\pi-2bt+2a \cos t)},$$

from which by giving proper values to θ and t we may find the c. g. of any portion of the surface symmetrical about the planes of i and j, and j and k.

We will next apply eq. (67) to the case of a hyperboloid of revolution of one sheet formed by revolving the straight line $_{\prime}\rho = _{\prime}\phi(t) = bj + t\varepsilon$ about the axis of i. For convenience we will suppose $S\varepsilon j = 0$, so that the generatrix in its initial position meets j at a distance b from the origin and is $\perp$ to j.

For the equation of the surface we have then $_{\prime}\rho = i^{th\div\pi}(bj+t\varepsilon)i^{-th\div\pi}$, i replacing the ε of eq. (67) $\therefore$ $_{\prime}\phi'(t) = \varepsilon$, $Vi_{\prime}\phi = Vi(bj + t\varepsilon) = bk + tVi\varepsilon$, $V_{\prime}\varphi_{\prime}\phi' = V(bj+t\varepsilon)\varepsilon = bj\varepsilon$, $S_{\prime}\phi_{\prime}\phi' Vi_{\prime}\phi = bSj\varepsilon(bk+tVi\varepsilon) = -b^2Si\varepsilon$, $Si_{\prime}\phi = tSi\varepsilon$.

$$\therefore\ _{\prime}\rho_0 = \frac{3}{4}\cdot\frac{1-u^4}{1-u^3}\cdot\int_{t_1}^{t_2}\left[\theta t i^{-1} Si\varepsilon - i^{\frac{2th}{\pi}}(bk+tVi\varepsilon)\right]_0^{th_1} dt \div \theta_1\int_{t_1}^{t_2} dt$$

$$= \frac{3}{4\theta_1(t_2-t_1)}\cdot\frac{1-u^4}{1-u^3}\left[\tfrac{1}{2}\theta_1(t_2^2-t_1^2)i^{-1}Si\varepsilon - \left(i^{\frac{2th_1}{}}-1\right)[bk(t_2-t_1)+\tfrac{1}{2}(t_2-t_1)Vi\varepsilon]\right]$$

$$= \frac{3}{4}\cdot\frac{1-u^4}{1-u^3}\left[\tfrac{1}{2}(t_2+t_1)i^{-1}Si\varepsilon - \theta_1^{-1}\left(i^{\frac{2th_1}{}}-1\right)[bk + \tfrac{1}{2}(i_2+t_1)Vi\varepsilon]\right].$$

Suppose $t_1 = 0$, $u = 0$ and $\theta_1 = \pi$, for a half revolution,

$$\therefore\ _{\prime}\rho_0 = \tfrac{3}{8}i^{-1}Si\varepsilon t_2 + \frac{3}{4\pi}Vi\varepsilon t_2 + \frac{3b}{2\pi}k.$$